

ON A QUESTION OF SADULLAEV CONCERNING BOUNDARY RELATIVE EXTREMAL FUNCTIONS

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ABSTRACT. We study the relation between certain alternative definitions of the boundary relative extremal function. For various domains we give an affirmative answer to the question of Sadullaev, [6], whether these extremal functions are equal.

1. INTRODUCTION

Let $D \subset \mathbb{C}^n$ be a smoothly bounded domain, $A \subset \partial D$, and let $\text{PSH}(D)^-$ stand for the family of non-positive plurisubharmonic functions on D . For $u \in \text{PSH}(D)^-$ as usual

$$u^*(z) = \limsup_{\zeta \rightarrow z, \zeta \in D} u(\zeta) \quad (z \in \overline{D}).$$

Sadullaev studied the first three of the following boundary extremal functions. For $z \in D$ consider

- (1) $\omega_1(z, A, D) = \omega^c(z, A, D) = \sup\{u(z), u \in \text{PSH}(D)^- \cap C(\overline{D}), u|_A \leq -1\}$,
- (2) $\omega_2(z, A, D) = \omega(z, A, D) = \sup\{u(z), u \in \text{PSH}(D)^-, u^*|_A \leq -1\}$,
- (3) $\omega_3(z, A, D) = \omega^n(z, A, D) = \sup\{u(z), u \in \text{PSH}(D)^-, \limsup_{z \rightarrow \zeta, z \in n_\zeta} u(z) \leq -1 \text{ for } \zeta \in A\}$ where n_ζ is the inward normal to ∂D at ζ ,
- (4) $\omega^R(z, A, D) = \sup\{u(z), u \in \text{PSH}(D)^-, \limsup_{r \rightarrow 1^-} u(rz) \leq -1, z \in A\}$, in case D is strongly star shaped

Actually, smoothness is needed only to define ω^n . It is clear that

$$\omega_1(\cdot, A, D) \leq \omega_2(\cdot, A, D) \leq \omega_3(\cdot, A, D).$$

This paper is motivated by the following question (Problem 27.4 in [6]): Suppose $A \subset \partial D$ is closed, for what i, j is $\omega_i^*(z, A, D) \equiv \omega_j^*(z, A, D)$?

The answer apparently depends on the geometry and convexity properties of D and the choice of the compact set $A \subset \partial D$. For instance we showed in [7] that Sadullaev's question has a positive answer when D is a smooth pseudoconvex Reinhardt domain and A is multi-circular. The result in [7] exploits the relation between relative extremal functions and convex functions in a Reinhardt domain.

In the present paper we answer in Section 3 the question affirmatively for ellipsoidal domains D_H , which are biholomorphic to the unit ball via a linear transformation. Here we exploit an idea of Wikström [1] and use Edwards duality theorem. In Section 4 we show equality for circular sets A in the boundary of circular, strongly star shaped domains D . We attempted to use the version of Edwards' theorem in [3] and found that their result is not correct. In the appendix we give two pertaining counterexamples.

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We denote the open unit disc in \mathbb{C} by \mathbb{D} , its boundary by \mathbb{T} and the unit ball in \mathbb{C}^n ($n \geq 2$) by \mathbb{B} . Some basic properties of the boundary relative extremal function are given in [6, 8, 9, 7]. Depending on the way the boundary is approached, plurisubharmonic function may have different boundary values. Wikström considered the compact set $A = \mathbb{T} \times \{0\}$ and the function $u \in \text{PSH}(\mathbb{B})$:

$$u(z) = \log \frac{|z_2|^2}{1 - |z_1|^2}.$$

He showed that $u^* = 0$ on A . The radial limit of u , $u^R = -\infty$ on A and the non-tangential limit of u , $u^\alpha = \log(1 - 1/2\alpha)$ on A [1, Example 5.5].

2. NOTATIONS AND DEFINITIONS

Let $D = \{\rho < 0\}$ be a domain in \mathbb{C}^n with C^1 -boundary and defining function ρ . For $z \in \overline{D}$ and $t \in \mathbb{R}$ let

$$n(z, t) = z - t\left(\frac{\partial \rho}{\partial \bar{z}_1}(z), \dots, \frac{\partial \rho}{\partial \bar{z}_n}(z)\right)$$

If $z \in \partial D$ the normal line n_z passing through z is parametrized by $\{n(z, t), t \in \mathbb{R}\}$.

Let $u : D \rightarrow \mathbb{R} \cup \{-\infty\}$ be bounded from above and $z \in \partial D$ we define u^n at z as

$$u^n(z) = \limsup_{t \downarrow 0} u \circ n(z, t).$$

Extend u^n to \overline{D} by setting $u^n(z) = u(z)$ if $z \in D$. Recall that D is called *strongly star shaped* if $r\overline{D} \subset D$ for $r \in]0, 1[$. If D is strongly star shaped, then for $z \in \partial D$ set $u^R(z) = \limsup_{r \uparrow 1} u(zr)$. Extend u^R to \overline{D} by setting $u^R(z) = u(z)$ if $z \in D$. Let $M(D)$ be the set of Borel probability measures with support on \overline{D} . For $z \in \overline{D}$ we consider four classes of positive measures

- (1) $J_z = \{\mu \in M(D) : u(z) \leq \int_{\overline{D}} u d\mu \text{ for all } u \in \text{PSH}(D)^- \cap \text{USC}(\overline{D})\}$
- (2) $J_z^c = \{\mu \in M(D) : u(z) \leq \int_{\overline{D}} u d\mu \text{ for all } u \in \text{PSH}(D)^- \cap C(\overline{D})\}$
- (3) $J_z^n = \{\mu \in M(D) : u^n(z) \leq \int_{\overline{D}} u^n d\mu \text{ for all } u \in \text{PSH}^-(D), \sup_{\overline{D}} u^n < \infty\}$
- (4) In case D is strongly star shaped, $J_z^R = \{\mu \in M(D) : u^R(z) \leq \int_{\overline{D}} u^R d\mu \text{ for all } u \in \text{PSH}^-(D), \sup_{\overline{D}} u^R < \infty\}$.

Clearly for $z \in D$ $J_z^n, J_z^R \subset J_z \subset J_z^c$. Wikström studied these classes and proved that $J = J^c = J^R$ on D if D is strongly star shaped, see [1, Proposition 5.4].

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3. APPLICATIONS OF WIKSTRÖM'S RESULTS

We use equalities between different classes of Jensen measures to prove the equivalence of different definitions.

Proposition 3.1. *Let $D \subset \mathbb{C}^n$ be a bounded domain with C^1 -boundary, $A \subset \partial D$ compact. If $J_z^c = J_z^n$ for all $z \in D$ then*

$$\omega^c(z, A, D) = \omega^n(z, A, D).$$

Proof. We know that $\omega^c(\cdot, A, D) \leq \omega^n(\cdot, A, D)$. Let us prove that $\omega^n(\cdot, A, D) \leq \omega^c(\cdot, A, D)$. Let u be in the family defining ω^n .

Set $g = -\chi_A$. Note that $u^n \leq g$ on \overline{D} . For $z \in D$ one has

$$u^n(z) \leq \inf \left\{ \int g d\mu, \mu \in J_z^n \right\} = \inf \left\{ \int g d\mu, \mu \in J_z^c \right\}, \text{ because } J_z^c = J_z^n.$$

Because g is lower semicontinuous on \overline{D} , Edwards' theorem (Corol. 2.2 in [1]) gives

$$u^n(z) \leq \inf \left\{ \int g d\mu, \mu \in J_z^c \right\} = \sup \{v(z), v \in \text{PSH}(D) \cap C(\overline{D}), v \leq g\} \leq \omega^c(z, A, D).$$

As u was taken arbitrarily in the family defining ω^n we infer that $\omega^n(z, A, D) \leq \omega^c(z, A, D)$ for all $z \in D$. Thus $\omega^c(\cdot, A, D) = \omega^n(\cdot, A, D)$. \square

The same proof applies to the next two propositions.

Proposition 3.2. *Let $D \subset \mathbb{C}^n$ be a bounded strongly star shaped domain and $A \subset \partial D$ compact. If $J_z = J_z^R$ for all $z \in D$ then*

$$\omega(z, A, D) = \omega^R(z, A, D).$$

Proposition 3.3. *Let $D \subset \mathbb{C}^n$ be a bounded domain and $A \subset \partial D$ compact. If $J_z = J_z^c$ for all $z \in D$ then $\omega(\cdot, A, D) = \omega^c(\cdot, A, D)$.*

Corollary 3.4. *If D is B -regular or if D is strongly star shaped with respect to the origin or if D is a polydisc then $\omega(\cdot, A, D) = \omega^c(\cdot, A, D)$.*

Proof. In these domains $J_z = J_z^c$ see [1, Thm.4.10, Thm.4.11, Cor. 4.3]. Then Proposition 3.3 gives the result. \square

For H a positive hermitian $n \times n$ -matrix, let $\rho_H(z) = \overline{z}^T H z$ on \mathbb{C}^n and set $D_H = \{z \in \mathbb{C}^n : \rho_H(z) < 1\}$.

Proposition 3.5. *On D_H we have $J_z^n = J_z^c$ for all $z \in D_H$.*

Proof. Set $D = D_H$. Let $z \in D$. Because for $u \in \text{PSH}(D) \cap C(\overline{D})$, $u = u^n$ on \overline{D} , we have $J_z^n \subset J_z^c$. Let $\mu \in J_z^c$ and $u \in \text{PSH}(D) \cap \text{USC}(\overline{D})$. Let $0 < r < 1$. Observe that in case of D_H the map $n(\cdot, r)$ is holomorphic and maps \overline{D} into D . Set $u_r(y) = u \circ n(y, r)$, $y \in \overline{D}$. Then u_r is plurisubharmonic in a neighborhood of \overline{D} , hence u_r can be approximated monotonically from above by functions in $\text{PSH}(D) \cap C(\overline{D})$. By the monotone convergence theorem $u_r(z) \leq \int u_r d\mu$ for all $r \in]0, 1[$. By Fatou's lemma

$$\limsup_{r \rightarrow 0} u_r(z) \leq \limsup_{r \rightarrow 0} \int_{\overline{D}} u_r(y) d\mu.$$

For $y \in D$ one has $\limsup_{r \rightarrow 0} u_r(y) = u^n(y)$. Because the set $[0, 1]$ is not thin at 0, see Theorem 2.7.2 in [4], we have

$$u^n(z) = u(z) = \limsup_{r \rightarrow 0} u_r(z) \leq \int_{\overline{D}} \limsup_{r \rightarrow 0} u_r(y) d\mu \leq \int_{\overline{D}} u^n(y) d\mu.$$

Thus $\mu \in J_z^n$ it follows that $J_z^c \subset J_z^n$. Hence $J_z^c = J_z^n$. \square

The unit ball, i.e. the case where $H = \text{Id}$, was done in [1]. Our proof is a slight modification of Wikström's.

Theorem 3.6. $\omega(., A, D_H) = \omega^n(., A, D_H) = \omega^R(., A, D_H) = \omega^c(., A, D_H)$ for all $A \subset \partial D_H$ compact.

Proof. By Proposition 3.5 $J^c = J^n$ and by Proposition 3.1 $\omega^c = \omega^n$. As D_H is strongly star shaped, $J = J^R$ see Prop. 5.4 in [1] and by Proposition 3.2 above the equality $\omega = \omega^R$ follows. \square

Remark that the theorem above holds for all positive normal matrix H .

4. CIRCULAR SETS

Our goal in this section is to generalize Theorem 2.11 in [7] and solve Sadullaev's problem for circular sets in circular, strongly star shaped, (hence balanced) domains.

Theorem 4.1. *Let D be a smooth circular, strongly star shaped domain and let $A \subset \partial D$ be a circular compact set. Then*

$$\omega^n(., A, D) \leq \omega^R(., A, D) = \omega^c(., A, D).$$

In particular,

$$\omega_1(z, A, D) = \omega_2(z, A, D) = \omega_3(z, A, D).$$

Proof. Let u be in the family defining $\omega^n(., A, D)$. Let ρ be a smooth defining function for D such that for all θ and y in a neighborhood of \overline{D} we have $\rho(y) = \rho(e^{i\theta}y)$. For $0 < t < 1$ consider the function

$$v_t(z, w) = u((n(w, t)z), \quad (w \in \overline{D}, z \in \mathbb{C}, |z| \leq 1).$$

For fixed t , w the function $v_t(., w)$ is subharmonic on the (closed) unit disc. Observe that $n(w, t)e^{i\theta} = n(e^{i\theta}w, t)$, so that for each $w \in A$ and all θ

$$\limsup_{t \downarrow 0} v_t(e^{i\theta}, w) \leq -1.$$

Hence for all $|z| \leq 1$, $\limsup_{t \downarrow 0} v_t(z, w) \leq -1$. It follows that $u(wz) \leq -1$ for $w \in A$ and $|z| \leq 1$. We infer that u belongs to the family defining $\omega^R(., A, D)$ and the inequality is proved.

Now suppose that u belongs to the family defining $\omega^R(., A, D)$. Then $u(wz) \leq -1$ for $w \in A$ and $|z| < 1$. Therefore, for $0 < r < 1$ $u_r(w) = u(rw)$ is a plurisubharmonic function in a neighborhood of \overline{D} that is less than -1 on A . Now u_r can be approximated from above on \overline{D} by a decreasing sequence $\{v_j\}$ of continuous PSH-functions. By Dini's theorem, for every $\epsilon > 0$ there is a j_0 so that $v_j \leq -1 + \epsilon$ on A hence also on a neighborhood of A . It follows that $u_r \leq \omega^c(., A, D)$, and then also $u \leq \omega^c(., A, D)$. \square

APPENDIX

We attempted to apply the non-compact version of Edwards' duality theorem stated in [3] to prove equalities for boundary extremal functions. However, we noticed that this versions of Edwards' theorem as stated, does not hold. This appendix contains some counterexamples.

Let $D \subset \mathbb{C}^n$ be a bounded set and $F \subset C(D)$ be a convex cone containing constants. $\mathbb{B}(D)$ denotes the set of Borel probability measures with support on D . For $z \in D$ set

$$J_z^F(D) = \{\mu \in \mathbb{B}(D), \text{ supp } \mu \subset D, u(z) \leq \int_D u d\mu \text{ for all } u \in F\}.$$

Let $g : D \rightarrow \mathbb{R}$ define

$$Sg(z) = \sup\{u(z), u \in F, u \leq g\}$$

and

$$Ig(z) = \inf\left\{\int_D g d\mu, \mu \in J_z^F(D)\right\}.$$

The following theorem is due to Edwards see ([2] Theorem 2.1).

Theorem 4.2 (Edwards). : Assume that D is compact and g is a bounded Borel function on D , then $Sg(z) \leq Ig(z)$. If g is lower semicontinuous, then $Sg = Ig$.

Edwards' theorem is very delicate. For instance if the kernel g is merely upper semicontinuous, the theorem may fail, see [2, 3]. We will show that the theorem may also fail if the set D is not compact, contrary to the following theorem, which was formulated in ([3, Thm.1.3]).

Theorem 4.3 ([3]). Let D be a locally compact Hausdorff space countable at infinity. If $g \in C(D)$ then either

$$Sg(z) = \inf\left\{\int_D g d\mu, \mu \in J_z^F(D)\right\}$$

or $Sg \equiv -\infty$.

However, this result does not hold if D is open.

Counterexample 4.4. For the sake of finding a contradiction, assume that Theorem 4.3 holds for all open set D' i.e

$$(1) \quad \sup\{u(z), u \in F, u \leq g\} = \inf\left\{\int_{D'} g d\mu, \mu \in J_z^F(D')\right\},$$

where $z \in D'$, $g \in C(D')$, $F \subset C(D')$ is a convex cone containing constants.

Let D be a bounded open ball and $V \subset\subset D$ be an open ball. Define

$$u_{D,V}(z) = \sup\{u(z), u \in \text{PSH}(D), u \leq -\chi_V\}.$$

Let $u \in \text{PSH}(D)^-$ so that the set $\{u = -\infty\}$ is dense in V . For $m > 0$ set $U_m = \{u/m < -1\} \cap V$, and $F = \text{PSH}(D) \cap C(\overline{D})$. Observe that the function $g_m = -\chi_{U_m}$ is continuous in the open set $D \setminus \partial U_m$ and that F is a convex cone in $C(D \setminus \partial U_m)$ containing the constants. By (1) we obtain in the open set $D \setminus \partial U_m$ the following equality (we take for D' the set $D \setminus \partial U_m$)

$$\inf\left\{\int_{D \setminus \partial U_m} g_m d\mu, \mu \in J_z^F(D \setminus \partial U_m)\right\} = \sup\{v, v \in F, v \leq g_m\} \text{ on } D \setminus \partial U_m.$$

If $v \in F$ and $v \leq g_m$, then $v \leq -\chi_V$ because $\overline{U_m} = \overline{V}$ implies $v \leq u_{D,V}$, hence

$$\inf\left\{\int_{D \setminus \partial U_m} g_m d\mu, \mu \in J_z^F(D \setminus \partial U_m)\right\} = \sup\{v, v \in F, v \leq g_m \text{ on } D \setminus \partial U_m\} \leq u_{D,V}.$$

As $J_z^F(D \setminus \partial U_m) \subset J_z^c$ we have on $D \setminus \partial U_m$

$$\inf\left\{\int_{\overline{D}} g_m d\mu, \mu \in J_z^c\right\} \leq \inf\left\{\int_{D \setminus \partial U_m} g_m d\mu, \mu \in J_z^F(D \setminus \partial U_m)\right\} \leq u_{D,V}.$$

Because D is a ball, by [1, Cor.4.3] $J_z = J_z^c$. It follows that

$$\inf \left\{ \int_{\overline{D}} g_m d\mu, \mu \in J_z \right\} = \inf \left\{ \int_D g_m d\mu, \mu \in J_z^c \right\} \leq u_{D,V} \text{ on } D \setminus \partial U_m.$$

Now u/m is plurisubharmonic and $u/m \leq g_m$, hence for all $m > 0$ one has

$$u/m(z) \leq \inf \left\{ \int_{\overline{D}} g_m d\mu, \mu \in J_z \right\} \leq u_{D,V}(z) \text{ for } z \in D \setminus \partial U_m.$$

As $D \setminus \overline{V} \subset D \setminus \partial U_m$ we have for all $m > 0$ that

$$\frac{u}{m} \leq u_{D,V} \text{ on } D \setminus \overline{V}.$$

This is impossible since

$$0 \equiv \left(\sup_m \frac{u}{m} \right)^* \leq u_{D,V} < 0 \text{ on } D \setminus \overline{V}.$$

The conclusion is that equality (1) is false in open sets D' .

Next we prove that the version of Edwards' theorem as stated in Theorem 4.2 does not hold for (open) B-regular domains.

Counterexample 4.5. Let D be a B-regular domain and $V \subset \partial D$ be relatively open. Then \overline{V} is not b-pluripolar, see Propositions 3.5 and 2.4 in [7]. There exists a countable $L \subset D$ so that $L \cup \overline{V}$ is compact in \overline{D} cf., [7, Lemma 4.3]. Set $g = -\chi_L$ and $F = \text{PSH}(D) \cap C(\overline{D})$. Note that g is lower semicontinuous in D and F is a cone in $C(D)$. If Theorem 4.2 would hold for D we would get

$$\begin{aligned} \inf \left\{ \int_D g d\mu, \mu \in J_z^F(D) \right\} &= \sup \{ u(z), u \in F, u \leq g \} \leq \omega(z, V, D), \\ \inf \left\{ \int_D g d\mu, \mu \in J_z^c \right\} &\leq \inf \left\{ \int_D g d\mu, \mu \in J_z^F(D) \right\} \leq \omega(., V, D) \end{aligned}$$

because $J_z^F(D) \subset J_z^c$,

$$\inf \left\{ \int_D g d\mu, \mu \in J_z \right\} = \inf \left\{ \int_D g d\mu, \mu \in J_z^c \right\} \leq \omega(., V, D)$$

because $J_z = J_z^c$,

$$\sup \{ u(z), u \in \text{PSH}(D), u \leq g \} \leq \inf \left\{ \int_D g d\mu, \mu \in J_z \right\} \leq \omega(., V, D).$$

Finally, because L is countable and therefore pluripolar, we would get

$$0 \equiv (\sup \{ u(z), u \in \text{PSH}(D), u \leq g \})^* \leq \omega(., V, D) \leq 0.$$

This is impossible since V is not b-pluripolar. The conclusion is that Edwards' theorem does not hold in D .

Remark 4.6. Approximating g by continuous functions one can show that Theorem 4.3 does not hold in B-regular domains. By domain we mean open and connected set.

These counterexamples make it unlikely that a useful non-compact version of Edwards' theorem can be found.

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